Computing mixed Nash equilibria

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Consider a game from Precept 3 with the following payoff matrix for two players A (his strategies are in the rows) and B (her strategies are in the columns):

	L	R
U	1,1	4,2
D	3,3	2,2

To find **pure Nash equilibria (NE)**, we check each of 4 pure strategies:

- Joint strategy (U, L): Player A has an incentive to switch to strategy D to get a payoff 3 instead of 1 - **not a NE**
- Joint strategy (U, R): Player A has no incentive to switch (he can only change it to (D, R) and his payoff will be worse, 2 instead of 4), Player B also has no incentive to switch (she can only change it to (U, L) and her payoff will be worse, 1 instead of 2) - **this is a NE**
- Joint strategy (D, L): **also a NE** (exercise: argue this similarly to the (U, R) case)
- Joint strategy (D, R): Player B has an incentive to switch to strategy L to get a payoff 3 instead of 2 - **not a NE**

To find **mixed Nash equilibria (mixed NE)**, let us first do it by definnition like we did in class for the Matching Pennies game. Then, we summarize a shorter approach that you discussed on the precepts.

Assume that the mixed joint strategy (p, q) was employed (Player A chooses U with probability p and Player B chooses L with probability q).

First, let us take the position of Player A. The payoff for Player A will be

$$pq \cdot 1 + p(1-q) \cdot 4 + q(1-p) \cdot 3 + (1-p)(1-q) \cdot 2 =$$
$$-4pq + 2p + q + 2 = \mathbf{p(2 - 4q) + q + 2}. \quad (1)$$

As Player A, we have control over the choice of p, but q is given for us. So, we reason about what would be our best response for various q.

- If $2 - 4q < 0$ (that is, $q > 1/2$), then the best p is minimal possible $p = 0$.
- If $2 - 4q > 0$ (that is, $q < 1/2$), then the best p is maximal possible $p = 1$.
- If $q = 1/2$, any choice for p gives the same payoff.

To see if any of these strategies – $(0, q)$ with $q > 1/2$, OR $(1, q)$ with $q < 1/2$, OR $(p, 1/2)$ for any p – contain equilibrium points, we need to check if Player B would also want to adhere to the same strategy from her side.

So, next step is to take the perspective of Player B. Her payoff is

$$pq \cdot 1 + p(1-q) \cdot 2 + q(1-p) \cdot 3 + (1-p)(1-q) \cdot 2 = \\ -2pq + q + 2 = \mathbf{q(-2p + 1) + 2}. \tag{2}$$

As Player B, we only control the choice of q, leading to the following reasoning for the different cases for the given p:

- If $1 - 2p > 0$ (that is, $p < 1/2$), then the best q is maximal possible $q = 1$.
- If $1 - 2p < 0$ (that is, $p > 1/2$), then the best q is minimal possible $q = 0$.
- If $p = 1/2$, any choice for q gives the same payoff[1].

So, Player B would support a joint strategy as long as it is $(p, 1)$ for $p < 1/2$, OR $(p, 0)$ for $p > 1/2$, OR $(1/2, q)$ for any q.

We now compare two conditions in blue and see which joint strategies (p, q) would satisfy both of them and get three options: $(0, 1)$ and $(1, 0)$ corresponding to the two pure Nash equilibria found above, and additionally a mixed one $(1/2, 1/2)$. The problem is solved now.

Remark 1. *1. Note that the second part of the reasoning that starts with the general joint strategy (p, q) finds **all** Nash equilibrium points (pure and mixed) and it is enough to solve the problem completely. The first part can be used to double-check all pure equilibrium points that were discovered in the general process.*

2. In the precept, you were shown a shorter way to find the same mixed equilibrium. It stems from the following observation: the only way a truly mixed strategy ($p \neq 0, 1$ or $q \neq 0, 1$) can be the best response of a player if all his/her response strategies are equally good (give the same payoff for that player). Why? The reason for this is that the payoff function is linear in p for fixed q (and vice versa, see equations (1),(2)), so, if it is not a constant function then the payoff is maximized in the endpoint of the segment $[0, 1]$. A player choosing p will choose $p = 0$ oor 1 then. ***So, this is the only case when a truly mixed strategy $p \in (0, 1)$ can be in a Nash equilibrium: if for all choices of p, the payoff of the respective player (who chooses p) is the same.***

[1]Note that most values of p result in optimal q being 0 or 1, i.e., a pure strategy is the best response. And only one option that allows for a mixed best response actually allows for any response. We explore this observation further in Remark 1.

Now, how can we use the remark to find (only a mixed) Nash equilibrium quicker? We conclude from the remark that if a (p, q) is a mixed equilibrium, then Player A can play the strategy p equally well as his strategies 0 and 1, that is, the strategies $(0, q)$, (p, q) and $(1, q)$ all give the same payoff for Player A. *We will actually use this fact to determine* q, namely

$$\text{payoff for A}(1, q) = \text{payoff for A}(0, q),$$

(note that we do not even consider the point (p, q) here in computation!)

$$q \cdot 1 + (1 - q) \cdot 4 = q \cdot 3 + (1 - q) \cdot 2$$

Solving this for q, we get $q = 1/2$.

We also conclude from the remark that if a (p, q) is a mixed equilibrium, then Player B can play the strategy q equally well as his strategies 0 and 1, that is, the strategies $(p, 0)$, (p, q) and $(p, 1)$ all give the same payoff for Player B. So,

$$\text{payoff for B}(p, 0) = \text{payoff for B}(p, 1),$$

and we use this to solve for p

$$p \cdot 1 + (1 - p) \cdot 3 = p \cdot 2 + (1 - p) \cdot 2$$

and we get $p = 1/2$.

Remark 2. *1. Any of two ways can be used to find Nash equilibria. Note that the second way does not give you pure NE, so they need to be found independently like on page 1.*

2. The same approach of indifference of payoff *at mixed equilibria can be employed for the games with more than wo strategies. See, for example, a Rock-Paper-Scissors game example in this tutorial Handout on Mixed Strategies by Ben Polak.*