Remarks on robust optimization

Robust optimization = with uncertainty in the data of the optimization problem.

Goal: make the best decision
- feasible for any input data, or
- feasible for the most cases

How to model uncertainty?

Case 1 (Non-random)

Assume that coefficients come from a particular set

<u>Example:</u> LP with polyhedral uncertainty

$$\begin{cases} \min c^T x \\ a_i^T x \le b_i \end{cases} \quad \text{where } a_i \in \mathcal{U}_{a_i}, \quad b_i \in \mathcal{U}_{b_i} \quad i = 1, \dots m$$

Note: with the trick $\begin{cases} \min\limits_x c^T x \\ \dots \end{cases} \iff \begin{cases} \max\limits_{t,x} t \\ c^T x \ge t \\ \dots \end{cases}$

we can assume there is no uncertainty in the objective function

If $b_i \in$ interval (1-dimensional) $b_i \in [b_i^f, b_i^c]$

$a_i^T x \le b_i$ for all admissible $b_i \iff a_i^T x \le b_i^f$

Now, $a_i \in \mathcal{U}_{a_i} := \{ a_i \mid \underbrace{D_i \cdot a_i \le d_i}_{\text{polyhedral uncertainty region}} \}$

Robust problem:

$$\begin{bmatrix} \min\limits_x c^T x \\ \text{s.t. } a_i^T x \le b_i^f \text{ for any } a_i : D_i \cdot a_i \le d_i \end{bmatrix} \iff$$

$$\begin{bmatrix} \min\limits_x c^T x \\ \text{s.t. } \begin{bmatrix} \max\limits_{a_i} a_i x \\ \text{s.t. } D_i a_i \le d_i \end{bmatrix} \le b_i^f \end{bmatrix} \iff$$

$$\left[\begin{array}{l} \min_x \; c^T x \\ \text{s.t.} \left[\begin{array}{l} \min_{p_i} \; p_i^T d_i \\ D_i^T p_i = x \\ p_i \geq 0 \end{array}\right] \leq b_i \end{array}\right] \iff$$

$$\left[\begin{array}{l} \min_{x, p_i} \; c^T x \\ p_i^T d_i \leq b_i \\ D_i^T p_i = x \\ p_i \geq 0 \end{array}\right]$$

This is an LP.

Objective function is the same, so take optimal x for one problem, need to check it is feasible for the other:

$\Leftarrow$ easy by definition of min

$\Rightarrow$ exists $p^\vee$: $D_i^T p_i = b_i$, $p_i^\vee \geq 0$, $p_i^{T\vee} d \leq b_i$

$x, p^\vee$ is a feasible pair for the second problem then.

Robust LP with ellipsoidal uncertainty $\to$ SOCP (HW)

Robust SOCP with ellipsoidal uncertainty $\to$ SDP (requires S-Lemma to prove)

$\mathcal{U}_{a_i}$ are ellipses

Robust SOCP with polyhedral uncertainty $\to$ NP-hard (1 linear uncertainty constraint – many linear constraints – S-Lemma is not tight any more..)

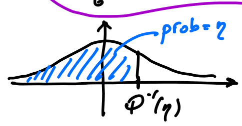
See Bertsimas, Brown, Caramanis [BBC] in general and links on its p. 12 in particular

Case 2 (Distributionally parametrized constraints)

Example: suppose in LP $a_i \sim N(\bar{a}_i, \Sigma)$

We want to impose that $\mathbb{P}(a_i^T x \leq b) \geq \eta$ constraints hold with probability at least η

$$\mathbb{P}(a_i^T x \leq b_i) = \mathbb{P}\left(\frac{a_i^T x - \bar{a}_i^T x}{\sigma} \leq \frac{b - \bar{a}_i^T x}{\sigma}\right) \geq \eta \iff \frac{b_i - \bar{a}_i^T x}{\sigma} \geq \Phi^{-1}(\eta)$$



Here, $\sigma^2 = \mathrm{var}(\bar{a}_i^T x) = x^T \Sigma x$

So, $\circ \iff$

$$\underbrace{\| \Sigma^{1/2} x \|_2 \leq \frac{b_i - \bar{a}_i^T x}{\varphi^{-1}(\eta)}}$$

This is SOCP

as long as $\varphi^{-1}(\eta) > 0$
or, equivalently, $\eta \geq \frac{1}{2}$
(probability to satisfy the constraint)

Case 3 (Robust stability)

Suppose we consider if a dynamical system $x_{k+1} = A x_k$ is GAS (globally asymptotically stable)

Suppose we have various estimates for A : $A_1, A_2, \dots A_m$

It is standard to model A as an element of a matrix set $\mathcal{A} := \operatorname{conv}\{A_1, \dots A_m\}$

Def If all matrices in $\mathcal{A}$ are stable then the system is robustly stable

Remark: it is not enough to have all $A_1, \dots A_m$ to be stable (see example on p7 of Lecture 16 [AAA])

$\rho(A_1) = 0.9887$
$\rho(A_2) = 0.9621$ but $\rho(\frac{3}{5}A_1 + \frac{2}{3}A_2) > 1$

$\rho(A)$ = spectral radius of A (= largest absolute value of eigenvalues)

note the difference with the spectral norm (= largest singular value) which is convex!

So, we need some uniform criterion for all matrices in the convex hull

One such sufficient condition is to have a common Lyapunov function:

$$\exists P : \quad A_i^T P A_i \prec P, \quad P \succ 0 \quad \forall i = 1 \dots m$$

Schur: $P - (PA_i)^T P (PA_i) \succ 0$

In this case for any $A = \sum \alpha_i A_i \quad \sum \alpha_i = 1 \quad \alpha_i > 0$

we have

$$\sum \alpha_i \cdot \left[\begin{array}{c|c} P & PA_i \\ \hline (PA_i)^T & P \end{array}\right] \succ 0$$

as PD matrices form a convex set

$$\Updownarrow$$

$$\left[\begin{array}{c|c} \sum \alpha_i P & \sum \alpha_i P A_i \\ \hline \sum \alpha_i \cdot A_i^T \cdot P & \sum \alpha_i P \end{array}\right] \overset{\sum \alpha_i = 1}{=} \left[\begin{array}{c|c} P & P \cdot A \\ \hline A^T P & P \end{array}\right] \succ 0$$

Note that if each A_i has its own P_i,

$$\sum \alpha_i P_i \cdot A_i \neq \underbrace{P}_{?} A$$